the distance $L(t_0)$ diminishes. For $\omega = 1$ when the characteristic dimension of the deformation pattern is commensurate with $\sqrt{lh}$, the quantity $L(t_0)$ is commensurate with l. But for $\omega = 2$ the characteristic dimension of the deformation pattern and the quantity $L(t_0)$ become commensurate with h. This also indicates that the dynamic processes corresponding to the values $\omega \geqslant 2$ are essentially three-dimensional in nature.

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HERTZ PROBLEM ON COMPRESSION OF ANISOTROPIC BODIES

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On the basis of results in [1], a derivation is given of the fundamental Hertz relationships for the compression of anisotropic (orthotropic) bodies which differs from [2]. It is shown that if the elastic constants satisfy some additional conditions, then the domain of contact is a circle in the compression of axisymmetric bodies along their common axes of geometric symmetry.

1. Formulation of the problem and its solution. Two bodies initially touching at a point and subjected to compressive forces P have a common elliptical contact area after deformation because of its smallness. If z_1 and z_2 are in the same direction as the internal normals to the surfaces bounding the bodies at the point of their initial contact, then the x, y axes in the common tangent plane can always be selected so that the equality

$$w_1 + w_2 = \delta - x^2 / 2R_1 - y^2 / 2R_2 \tag{1.1}$$

would hold in the contact domain. Here w_j are elastic displacements of the body points in the z_j direction, δ is the approach of the bodies, R_j are specified and determined by the shape of the bodies in the neighborhood of their initial contact point [1]. The pressure domains of the bodies are replaced by half-spaces in the computation of w_j because of the smallness of the dimensions. Therefore, in conformity with [1], the stress on the pressure area is determined by

$$\sigma_z = 3P\,(2\pi ab)^{-1}\,(1 - x^2 / a^2 - y^2 / b^2) \tag{1.2}$$

in the absence of friction. There are no stresses on the half-space boundaries outside this area.

The corresponding loading function is [1]

$$\Psi(\Omega_{kj}) = \frac{3P}{8\pi (ab)^{3/2}}\left[-\sqrt{ab}\,\Omega_{kj} + \frac{ab-\Omega_k^2}{2}\ln\frac{\Omega_{kj}-\sqrt{ab}}{\Omega_{kj}+\sqrt{ab}}\right] \qquad (1.3)$$

$$\Omega_{kj} = (\alpha x + \beta y + \nu_{kj} z)\,\Delta^{-1}, \quad \alpha = \cos\theta, \ \beta = \sin\theta$$

Taking into account its limit value for $z_j = 0$ and the choice of the logarithm branches mentioned in [1], we obtain

$$w_j = \frac{3P}{4\pi (ab)^{3/2}}\int_0^{\pi/2} \mathrm{Re}\sum_{k=1}^{3} i\,\frac{\Delta_{kj}^{(3)}\Delta_{kj}}{\Delta_{0j}}\left(ab - \frac{x^2\alpha^2+y^2\beta^2}{\Delta^2}\right)\frac{d\theta}{\Delta} \qquad (1.4)$$

The values of $\Delta_{kj}^{(3)}$, Δ_{kj}, Δ_{0j}, Δ are indicated in [1]. Substituting into (1.1) and equating coefficients of identical powers, we derive the fundamental relations

$$\frac{3P}{4\pi (ab)^{1/2}}\int_0^{\pi/2}\sum_{j=1}^{2}\sum_{k=1}^{3}\mathrm{Re}\, i\,\frac{\Delta_{kj}^{(3)}\Delta_{kj}}{\Delta_{0j}}\frac{d\theta}{\Delta} = \delta \qquad (1.5)$$

$$\frac{3P}{2\pi (ab)^{1/2}}\int_0^{\pi/2}\sum_{j=1}^{2}\sum_{k=1}^{3}\mathrm{Re}\, i\,\frac{\Delta_{kj}^{(3)}\Delta_{kj}}{\Delta_{0j}}\,\alpha_\rho^2\,\frac{d\theta}{\Delta} = R_\rho^{-1} \quad (\rho = 1, 2) \qquad (1.6)$$

$$\alpha_1 = \alpha, \ \alpha_2 = \beta$$

The equalities (1.5) and (1.6) permit finding the semi-axes a, b of the elliptical contact area and the approach δ of the bodies. In fact, if ε is the eccentricity of the desired ellipse, then $\Delta = [(a/b)(1-\beta^2\varepsilon^2)]^{1/2}$ and we derive from (1.6)

$$\int_0^{\pi/2}\sum_{j=1}^{2}\sum_{k=1}^{3}\mathrm{Re}\, i\Delta_{kj}^{(3)}\Delta_{kj}\Delta_{0j}^{-1}\left(\alpha^2 - \frac{R_2\beta^2}{R_1}\right)(1-\beta^2\varepsilon^2)^{-3/2}d\theta = 0 \qquad (1.7)$$

$$\left(\frac{3}{4\pi}\right)Pa^{-3}\int_0^{\pi/2}\sum_{j=1}^{2}\sum_{k=1}^{3}\mathrm{Re}\, i\Delta_{kj}^{(3)}\Delta_{kj}\Delta_{0j}^{-1}(1-\beta^2\varepsilon^2)^{-3/2}d\theta = R_1^{-1} + R_2^{-1} \qquad (1.8)$$

Determining ε from (1.7) and substituting into (1.8), we find a and then b, δ. It is easy to verify that (1.5) – (1.8) go over into the appropriate Hertz relationships for an isotropic medium.

2. Particular case of an orthotropic body. In the general case of an orthotropic body, $b \neq a$ during compression of axisymmetric bodies along their common axes of geometric symmetry, as is seen from (1.6), i.e. the pressure domain has the shape of an ellipse. However, it is circular if the elastic constants of the media satisfy the conditions

$$B_j = A_j, \ M_i = L_j, \ G_j = F_j \ (j = 1, 2) \qquad (2.1)$$

In fact, (1.3) in [1] can be written under the conditions (2.1) as (we limit ourselves to an analysis of the relationships for one medium by omitting the subscript) (2.2)

$$\sum_{k=1}^{3}(\nu_k^5 + L\nu_k^3 + E\nu_k)\,\omega_k = \frac{\Psi^+}{CL^2} \qquad \sum_{k=1}^{3}(\nu_k^4 - D)\,\omega_k = 0, \qquad \sum_{k=1}^{3}\nu_k^2\omega_k = 0$$

$$E = (1 / CL^2) \{N [AC - F (L + F)] + K_0 [C (A + H) - 2F (L + F)] \alpha^2\beta^2\}$$
$$L = (1 / CL) [C (A + N) - F (L + F)] \tag{2.3}$$
$$D = (1 / LF) [AN + K_0 (A + H) \alpha^2\beta^2], \quad K_0 = A - 2N - H$$

Here ν_k are the roots of Eqs. (1.2) from paper [1]. Reducing the fraction by $(\nu_1 - \nu_2)\times(\nu_2 - \nu_3)(\nu_3 - \nu_1)$, we deduce from (2.2)

$$w(x, y, 0) = \int_0^{2\pi} \operatorname{Re} i \frac{\Delta_1^*}{\Delta_0^*} d\theta \tag{2.4}$$

Here

$$\Delta_1^* = -(\nu_1 + \nu_2)(\nu_2 + \nu_3)(\nu_3 + \nu_1)(L + F) / CL \tag{2.5}$$

$$\Delta_0^* = \begin{vmatrix} S_1(S_2 + L) + \Pi & \nu_2^5 + L\nu_2^3 + E\nu_2 & A_{23} \\ S_2 + S_{ik} & \nu_2^4 - D & (\nu_2 + \nu_3)(\nu_2^2 + \nu_3^2) \\ 1 & \nu_2^2 & \nu_2 + \nu_3 \end{vmatrix} \tag{2.6}$$

$$S_1 = \nu_1 + \nu_2 + \nu_3, \quad \Pi = \nu_1\nu_2\nu_3 \tag{2.7}$$
$$S_{ik} = \nu_1\nu_2 + \nu_2\nu_3 + \nu_3\nu_1, \quad S_2 = \nu_1^2 + \nu_2^2 + \nu_3^2$$
$$A_{23} = \nu_3^4 + \nu_3^3\nu_2 + \nu_3^2\nu_2^2 + \nu_3\nu_2^3 + \nu_2^4 + L(\nu_3^2 + \nu_3\nu_2 + \nu_2^2) + E$$

Expanding the determinant (2.6), we obtain

$$\Delta_0^* = [D (S_2 + L) + \Pi^2] S_{ik} - (E + L) \Pi S_1 + E - \nu_1^2\nu_2^2 - \nu_2^2\nu_3^2 - \nu_3^2\nu_1^2) D + L\Pi^2 \tag{2.8}$$

It is easy to verify that $D (S_2 + L) + \Pi^2 = 0$. Furthermore, we have

$$\nu_1^2\nu_2^2 + \nu_2^2\nu_3^2 + \nu_3^2\nu_1^2 = (1 / CL^2) \{N [AC - (L + F)^2] + L^2 (A + N) + K_0 [C (A + H) - 2 (L + F)^2] \alpha^2\beta^2\} \tag{2.9}$$
$$\nu_1^2\nu_2^2\nu_3^2 = -FD / C \tag{2.10}$$

Let us consider $S_1 = \nu_1 + \nu_2 + \nu_3$. Here the ν_k are understood to be the roots of (1.2) from [1] with positive imaginary part for all α, β. Their existence is assured by the requirement of total ellipticity of the equilibrium equations of the anisotropic media under consideration (otherwise, the real characteristics of these equations are easily constructed).

It follows from (2.10) that there are two possibilities: (a) all the ν_k^2 are negative reals (in this case all the ν_k are pure imaginary), (b) one of the ν_k^2, say ν_1^2 is a negative real quantity and the rest are complex conjugate. Here $\nu_1 = in_1$, $n_1 > 0$; $\nu_2 = m + in_0$, $\nu_3 = -m + in_0$, $n_0 > 0$. In both cases we have: $S_1 = \text{in}$, $n > 0$. From (2.10) we deduce

$$\Pi = -i\sqrt{FD / C} \tag{2.11}$$

The minus sign should be taken in front of the square root since $\nu_1 = \nu_2 = \nu_3 = i$, $\Pi = -i$ in the case of isotropic medium. Taking into account the values of the quantities mentioned, we obtain

$$-\Delta_0^* = (L+F)\,/\,CL\,[(AC-F^2)\,/\,C - 2K_0\alpha^2\beta^2]\,D + \qquad (2.12)$$
$$(1\,/\,LF)\,\{N\,(AC-F^2) + K_0\,[(A+H)\,C - 2F^2]\,\alpha^2\beta^2\}\,\sqrt{FD\,/\,Cn}$$

If the elastic constants are positive and subject to the inequalities

$$HC - F^2 \geqslant 0, \quad A \geqslant H \qquad (2.13)$$

then the right side of (2.12) is positive. The conditions (2.13) are satisfied for all real anisotropic bodies of the class under consideration, which are presented in [3]. Furthermore, we have $\Delta_1^* = i\Delta_1^{**}$, where $\Delta_1^{**} > 0$ in both the above-mentioned cases. Therefore, assuming $\Delta_0^* = -\Delta_0^{**}$, $\Delta_2^{**} > 0$, we obtain for all α, β

$$\sum_{k=1}^{3} \operatorname{Re} i\Delta_k^{(3)}\Delta_k\Delta_0^{-1} = \frac{\Delta_1^{**}}{\Delta_0^{**}} > 0 \qquad (2.14)$$

It is seen that the expression (2.14) depends only on α^2, β^2 and is symmetric relative to these arguments. Thus, Eq. (1.7) is written as

$$\int_0^{\pi/2} T(\alpha^2, \beta^2)\,(\alpha^2 - R_2R_1^{-1}\beta^2)\,(1 - \beta^2\varepsilon^2)^{-3/2}\,d\theta = 0 \qquad (2.15)$$

$$T(\alpha^2, \beta^2) = \sum_{j=1}^{2}\sum_{k=1}^{3} \operatorname{Re} i\Delta_{kj}^{(3)}\Delta_{kj}\Delta_{0j}^{-1} > 0, \quad T(\alpha^2, \beta^2) = T(\beta^2, \alpha^2)$$

for media satisfying the conditions (2.1).

Let us prove the following result. If $R_1 = R_2 = R$, then $\varepsilon = 0$. In fact, in this case we have

$$\int_0^{\pi/2} T(\alpha^2, \beta^2)\,(\alpha^2 - \beta^2)\,(1 - \beta^2\varepsilon^2)^{-3/2}\,d\theta = 0 \qquad (2.16)$$

Hence, changing the variable of integration and assuming $\theta_1 = \theta - \pi\,/\,2$, we deduce

$$\int_0^{\pi/2} T(\alpha^2, \beta^2)\,(\alpha^2 - \beta^2)\,(1 - \alpha^2\varepsilon^2)^{-3/2}\,d\theta = 0 \qquad (2.17)$$

Subtracting (2.17) from (2.16), we obtain

$$\varepsilon^2 \int_0^{\pi/2} T_1\,(\alpha^2 - \beta^2)^2\,d\theta = 0 \qquad (2.18)$$

Here

$$T_1 = T\,[2 - \varepsilon^2 + (E_1E_2)^{1/2}]\,(E_1E_2)^{-3/2}\,[E_1^{1/2} + E_2^{1/2}]$$

$$E_1 = 1 - \alpha^2\varepsilon^2, \quad E_2 = 1 - \beta^2\varepsilon^2$$

Since $T_1 > 0$, the integral in (2.18) is then different from zero and, consequently, $\varepsilon = 0$, q.e.d.

Let us introduce the notation

$$\int_0^{\pi/2} T(\alpha^2, \beta^2)\,d\theta = m_0 = m_1^\circ + m_2^\circ, \quad m_j^\circ = 2\sum_{k=1}^{3} \operatorname{Re} i\Delta_{kj}^{(3)}\Delta_{kj}\Delta_{0j}^{-1}$$

then

$$\int_0^{\pi/2} T(\alpha^2, \beta^2)\,\alpha^2 d\theta = \int_0^{\pi/2} T(\alpha^2, \beta^2)\,\beta^2 d\theta = \frac{1}{4}\,m_0$$

From (1.6) we derive formulas for the radius of the contact circle a and the approach of the bodies δ

$$a = [3\,(8\pi)^{-1} RPm_0]^{1/3}, \quad \delta = [3\,(8\pi)^{-1} PR^{1/2} m_0]^{2/3} \qquad (2.19)$$

According to (1.4), the elastic displacements of points of the bodies on the contact section are determined by the formulas

$$w_j = m_j{}^0 m_0{}^{-1}\,(\delta - \rho^2 / 2R), \quad \rho^2 = x^2 + y^2 \qquad (2.20)$$

The relationships (2.19), (2.20) differ from the corresponding Hertz formulas for an isotropic medium just by the value of the constants m_0, $m_j{}^0$.

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CONTACT PROBLEM FOR A PLANE CONTAINING A SLIT OF VARIABLE WIDTH

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The problem of compression of an elastic plane with a slit of variable width commensurate to the elastic strains is considered. The case of the origination of several contact sections of the slit edges is investigated. Adhesion of the edges hence occurs at some part of the contact area, while slip is possible at the rest of this area. A solution of the problem is obtained in quadratures by the Muskhelishvili method using the apparatus of linear conjugates of analytic functions. The stress and displacement potentials are found, the magnitudes of the contact sections and the adhesion zones are determined. A specific example is analyzed and numerical computations are carried out.

The contact problem for a plane weakened by a constant-width rectilinear slit has been considered in [1 – 3].

1. An infinite elastic isotropic plane is weakened by a variable width rectilinear slit $h\,(x)$ commensureate with the elastic strains. The plane is compressed by uniformly distributed stress resultants with components P and T (Fig. 1), applied at infinity. The slit edges make contact along the sections (α_k, β_k) during deformation. Each contact area consists of an adhesion section of the edges (c_k, d_k) and two sections (α_k, c_k) and (d_k, β_k) on which slip is possible.

Let us use the notation: L_1 is the set of adhesion sections, L_2 is the set of slip sections,